PDE 1 Sheet 2

1. $\Omega = \mathbb{R}^n \times (0,\infty)$ $g \in L^\infty(\mathbb{R}^n)$, $b \in \mathbb{R}^n$.

Consider

$$(*) \begin{cases} u_t + b \cdot D_x u = 0 & \text{in } \Omega \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

When g is smooth: $u(x,t) = g(x-tb)$ solves $(*)$.
However, here g is not differentiable. However, we can still consider <u>weak solutions</u> to $(*)$.

Idea behind weak solutions: take $\varphi \in C_c^\infty(\Omega)$, ~~and integrate~~ multiply differential equation with φ; integrate over Ω:

if u smooth solves $(*)$, $\int_\Omega (u_t + b \cdot D_x u)\varphi = 0$ (2)

Integrate by parts / Gauss-Green:

$$\int_{\partial B} u(x,t)\,\varphi(x,t)\,\nu^{(i)}\,d(x,t) = 0 \quad 1 \le i \le n+1$$

when $\varphi = 0$ outside ball $B \subseteq \mathbb{R}^{n+1}$!

$$\shortparallel$$

$1 \le i \le n$ $\int_B \frac{\partial u}{\partial x_i}\varphi\,d(x,t) + \int_B \frac{\partial \varphi}{\partial x_i} u\,d(x,t)$

$$\shortparallel$$

$$\int_B u_t \varphi\,d(x,t) + \int_B \varphi_t u\,d(x,t)$$

So $\int_\Omega \frac{\partial u}{\partial x_i}\varphi = -\int_\Omega \frac{\partial \varphi}{\partial x_i} u$ ($\varphi = 0$ ~~outside~~ on $\Omega \setminus B$!)

$\int_\Omega u_t \varphi = -\int_\Omega \varphi_t u$

taking derivatives on φ instead

Hence, if u satisfies (2), then also $\int_\Omega (\varphi_t + b D_x \varphi(x,t))\, u\, dx = 0$

This expression makes sense even if u is not differentiable.

We can look for solutions to this instead!

Show: $u(x,t) = g(x-tb)$ is a weak solution to $(*)$.

$$\int_\Omega (\varphi_t(x,t) + b \cdot D_x \varphi(x,t))\, g(x-tb)\, d(x,t)$$

substitution $y = x - tb$. $x \in \mathbb{R}^n$, so $y \in \mathbb{R}^n$ too. Jacobian of transformation is 1.

$$\int_\Omega \varphi_t(y+tb,t) + b \cdot D_x \varphi(y+tb,t)\, g(y)\, d(y,t)$$

$$\begin{pmatrix} x \\ t \end{pmatrix} \to \begin{pmatrix} y \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} I_n & b \\ 0 & 1 \end{pmatrix}}_{\det = 1!} \begin{pmatrix} x \\ t \end{pmatrix}$$

~~Now note that~~ $\cancel{\frac{\partial \varphi}{\partial t}(y+tb,t)} = \cancel{\frac{\partial}{\partial t}(\varphi(y+tb,t))}$

$=$

Write $\psi(y,t) = \varphi(y + tb, t)$ $\quad (y,t) \in \Omega = \mathbb{R}^n \times (0,\infty)$

Then note that $\frac{\partial \psi}{\partial y_i}(y,t) = \frac{\partial \varphi}{\partial x_i}(y+tb,t)$ So $D_y \psi(y,t) = D_x \varphi(x,t)$

$$\frac{\partial \psi}{\partial t}(y,t) = \frac{\partial \varphi}{\partial t}(y+tb,t) + b \cdot D_x \varphi(y+tb,t)$$

Here we get

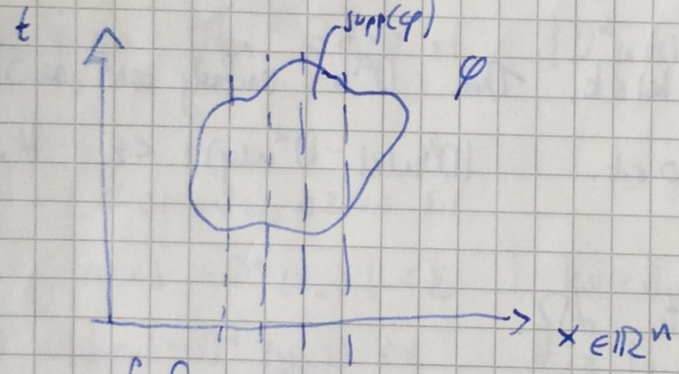
$$\int_\Omega \left[\varphi_t(y+tb,t) + b \cdot D_x \varphi(y+tb,t)\right] g(y)\, dy$$

$$= \int_\Omega \psi_t(y,t) - b\cdot D_x\varphi(y+tb,t) + b\cdot D_{xy}\psi(y,t)\Big]\, g(y)\, d(y,t)$$

$$= \int_\Omega \psi_t(y,t) - b\cdot D_x\varphi(y+bt,t) + b\cdot D_x\varphi(\cancel{x}\, y+bt,t)\Big]\, g(y)\, d(y,t)$$

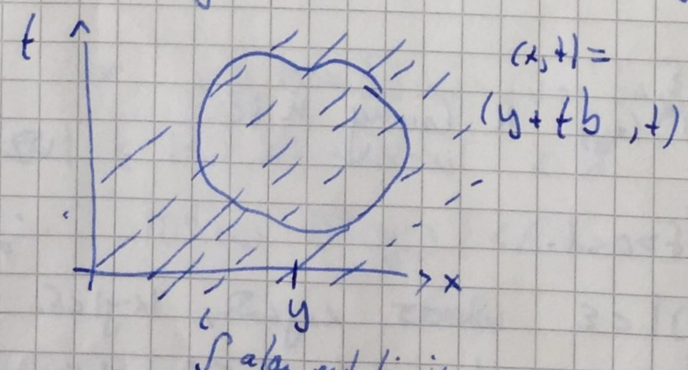
$$= \int_\Omega \psi_t(y,t)\, g(y)\, d(y,t)$$

$$= \int_{\mathbb{R}^n} \int_0^\infty \underbrace{\tfrac{\partial}{\partial t}\big(\varphi(x+tb,t)\big)}_{=0 \text{ as } \varphi \text{ has compact support!}} dt\; g(y)\, dy \quad = 0.$$



$\int_x \int_t$

if $u(x,t) = g(x-tb)$, u is constant along these lines



$\int$ along each line is 0.

(2) $\Omega \subset \mathbb{R}^n$ open, bounded

$$C^k(\bar{\Omega}) := \{ u \in C^k(\Omega) \mid C^\alpha u \text{ has a cont. extension on } \bar{\Omega} \ \forall \alpha \in \mathbb{N}_0^n, |\alpha| \le k \}$$

(a) $u \in C^k(\bar{\Omega})$

(b) $u \in C^k(\Omega)$ and $D^\alpha u$ uniformly continuous on Ω $\forall \alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$.

<u>(a) => (b):</u> ~~ie~~ for $|\alpha| \le k$, $D^\alpha u$ can be continuously extended to $\bar{\Omega}$.

$\bar{\Omega}$ closed bounded set, $D^\alpha u \in C(\bar{\Omega})$. So $D^\alpha u$ uniformly continuous on Ω (Analysis II).

<u>(b) => (a):</u> Let $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le k$. Then $D^\alpha u$ uniformly cont on Ω.

ie $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|D^\alpha u(x) - D^\alpha u(y)| < \varepsilon \ \forall x, y \in \Omega$ with $|x-y| < \delta$.

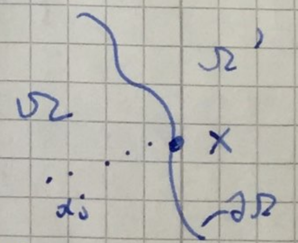
(one δ works $\forall \varepsilon$).

We need to extend $D^\alpha u$ to $\partial\Omega$.

Let $x \in \partial\Omega$. Then $\exists (x_j) \subset \Omega$ s.t. $x_j \to x$ in $\mathbb{R}^n$

(take $r_j \searrow 0$ then $\forall j \ \exists x_j \in B(x, r_j) \cap \Omega$).

Note that ~~a~~ $y_j := D^\alpha(x_j)$ is Cauchy in $\mathbb{R}$



Why? Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t.

$|D^\alpha u(x) - D^\alpha u(y)| < \varepsilon$ whenever $x, y \in \Omega$, $|x-y| < \delta$.

Since (x_j) is a convergent sequence in $\mathbb{R}^n$, it is also Cauchy.

i.e. $\exists j_0 \in \mathbb{N}$ s.t. $|x_j - x_k| < \delta \quad \forall j, k \ge j_0$.

Then $|D^\alpha u(x_j) - D^\alpha u(x_k)| < \varepsilon \quad \forall j, k \ge j_0$.

So $(D^\alpha(x_j))$ is Cauchy in $\mathbb{R}$.

Hence $\exists$ $a(x) = \lim_{j \to \infty} D^\alpha(x_j)$.

This can be done for every $x \in \partial\Omega$.

Need to show:

(1) $a(x)$ well-defined (independent of choice of $x_j \to x$)

(2) $a(x)$ continuously extends $D^\alpha u$ onto $\overline{\Omega}$.

ad(1): Suppose $x \in \partial\Omega$ and (x_j), (y_j) are 2 different sequences in Ω converging to x.

Write $a(x) = \lim_{j \to \infty} D^\alpha(x_j)$, $b(x) := \lim_{j \to \infty} D^\alpha(y_j)$.

Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. $|D^\alpha u(x) - D^\alpha u(y)| < \varepsilon$ whenever $|x-y| < \delta$, $x, y \in \Omega$.

Also $\exists j_1, j_2, j_3, j_4$ s.t.

$|a(x) - D^\alpha(x_j)| < \varepsilon \quad \forall j \geq j_1$

$|b(x) - D^\alpha(y_j)| < \varepsilon \quad \forall j \geq j_2$

$|x_j - x| < \delta/2 \quad \forall j \geq j_3$

$|y_j - x| < \delta/2 \quad \forall j \geq j_4$

Then for $j \geq \max\{j_1, j_2, j_3, j_4\}$

$|x_j - y_j| \leq |x_j - x| + |x - y_j| < \delta$

so $|D^\alpha(x_j) - D^\alpha(y_j)| < \varepsilon$

And $|a(x) - b(x)| \leq |a(x) - D^\alpha(x_j)| + |D^\alpha(x_j) - D^\alpha(y_j)| + |D^\alpha(y_j) - b(x)|$

$< 3\varepsilon$

So $a(x) = b(x)$. Extension $a(x)$ of $D^\alpha u$ to $\partial\Omega$ well defined.

ad (2) : By assumption $D^\alpha u$ cont on interior Ω.

If $x \in \partial\Omega$, $\varepsilon > 0$, $a(x) = \lim_j D^\alpha u(x_j)$, $x_j \in \Omega$, $x_j \to x$.

$\exists \delta > 0$ s.t. $|D^\alpha u(x) - D^\alpha u(y)| < \varepsilon$ $\forall x, y \in \Omega$, $|x-y| < \delta$.

Let $y \in \bar{\Omega}$, ~~$|x-y| < \delta/2$~~, 2 cases:

Case 1 : $y \in \Omega$. Then if $|x-y| < \delta/2$,

Then $|a(x) - D^\alpha u(y)| \leq \underbrace{|a(x) - D^\alpha u(x_j)|}_{< \varepsilon \ j \text{ large enough}} + \underbrace{|D^\alpha u(x_j) - y|}_{\leq \varepsilon \text{ if } |x_j - x| < \delta/2}$

(then $|x_j - y| < \delta$)

Case 2: $y \in \partial\Omega$. Then

$$|a(x) - a(y)| \leq \cancel{|D^\alpha(x_j)} \underbrace{|a(x) - D^\alpha u(x_j)|}_{A} + |D^\alpha u(x_j) - D^\alpha u(y_j)| + \underbrace{|D^\alpha u(y_j) - a(y)|}_{B}$$

Choose j large enough so that $A, B < \varepsilon$

and $|x_j - x|$, $|y_j - y| < \delta/2$

So $|x_j - y_j| < \delta$. ✓

(3) $u \in L^1(\mathbb{R}^n)$, $\varphi \in C_c^1(\mathbb{R}^n)$

$$(\varphi * u)(x) := \int_{\mathbb{R}^n} \varphi(x-y)\, u(y)\, dy$$

Show $\varphi * u \in C^1(\mathbb{R}^n)$ with

$$\frac{\partial}{\partial x_i}(\varphi * u)(x) = \int_{\mathbb{R}^n} \frac{\partial \varphi}{\partial x_i}(x-y)\, u(y)\, dy$$

Pf: Let $h \neq 0$, Fix $x \in \mathbb{R}^n$, $1 \leq i \leq n$. e_i - i^{th} coord. vector

$$\frac{(\varphi * u)(x+he_i) - (\varphi * u)(x)}{h} = \int_{\mathbb{R}^n} \underbrace{\frac{\varphi(x+he_i-y) - \varphi(x-y)}{h}\, u(y)}_{= v_h(y)}\, dy$$

$\varphi \in C_c^1(\mathbb{R}^n)$, so integrand is 0 outside some bounded set

(y $\text{supp}(\varphi) = K$, $\text{supp}(\varphi(x-\cdot)) = \{y \in \mathbb{R}^n, x-y \in K\}$ $= x - K$.)

$\nabla\varphi(x-\cdot)$ is also bounded.

Let $M := \max_{x \in \mathbb{R}^n} |\nabla\varphi(x)|$. $(= \|\nabla\varphi\|_\infty)$

Then for $y_1, y_2 \in \mathbb{R}^n$, $|\varphi(y_1) - \varphi(y_2)| \leq M|y_1 - y_2|$

(φ is Lipschitz continuous, as it has bounded derivative)

$v_h(y) \longrightarrow \frac{\partial \varphi}{\partial x_i}(x-y)\, u(y)$ point-wise as $h \to 0$.

Also, $|v_h(y)| \leq M|u(y)| \in L^1(\mathbb{R}^n)$

Hence we can use the DCT to take the limit outside the integral. i.e.

$$\exists \lim_{h \to 0} \frac{(\varphi * u)(x + he_i) - (\varphi * u)(x)}{h} = \left(\frac{\partial \varphi}{\partial x_i} * u\right)(x)$$

Show $\left(\frac{\partial \varphi}{\partial x_i} * u\right)$ cont: let $z_j \to x$ in $\mathbb{R}^n$.

Then $\left(\frac{\partial \varphi}{\partial x_i} * u\right)(z_j) = \int_{\mathbb{R}^n} \underbrace{\frac{\partial \varphi}{\partial x_i}(z_j - y)\, u(y)}_{\to \frac{\partial \varphi}{\partial x_i}(x-y) u(y)} \, dy$

Also, $\left|\frac{\partial \varphi}{\partial x_i}(z_j - y)\, u(y)\right| \leq M|u(y)| \in L^1(\mathbb{R}^n)$

apply DCT.